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Comparison and Oscillation Theorems for Equations with Middle Terms of Order $n - 1$

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In this paper, we study differential equations of the form

$$x^{(n)} + P_1(t, x, x', \ldots, x^{(n-1)})\, x^{(n-1)} + H_1(t, x) = 0, \qquad n \text{ even}. \tag{*}$$

We first show that the existence of a positive solution of (*) implies the same fact for the equation

$$x^{(n)} + P_2(t)\, x^{(n-1)} + H_2(t, x) = 0, \tag{**}$$

provided that the pairs P_1, P_2 and H_1, H_2 are properly related. These existence results are then used to show that the oscillation of (**) implies that of (*).

Applications of these considerations are given for certain rather general second-order equations which include as special cases generalized forms of the equation of Lienard and the equation of Van der Pol. We also consider certain forced equations with middle terms and apply the methods developed in the proofs of the main results (Theorem 2.1 and Theorem 2.4), to ensure the oscillation or the convergence to zero of all of their solutions. The reader is referred to Svec [10] and Bobisud [3] for some recent results concerning comparison theorems between second-order equations with middle terms. Oscillation results for such equations, but without comparison methods, have also been given by Bobisud [1, 2] and Erbe [4]. A general comparison theorem for nth-order equations without middle terms was given by the first of the authors in [5] and the present paper contains several extensions of and overlappings with that theorem. For a survey article on nonlinear oscillation in the present spirit the reader is referred to [6].

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1. Preliminaries

In what follows, n is even, $R = (-\infty, \infty)$, $R_+ = [0, \infty)$, and $R_- = (-\infty, 0]$. Consider the differential equation

$$x^{(n)} + F(t, x, x', \ldots, x^{(n-1)}) = 0, \tag{1.1}$$

where $F: R_+ \times R^n \to R$ is continuous. Then by a "solution" of (1.1) we mean any function $x \in C^n[t_x, \infty)$ which satisfies (1.1) on $[t_x, \infty)$. The number t_x depends on x. A solution $x(t)$, $t \geqslant t_x$ of (1.1) is "oscillatory," if there is an unbounded set of zeros of $x(t)$ on $[t_x, \infty)$. Equation (1.1) is "oscillatory" ("B-oscillatory") if all of its solutions (bounded solutions) are oscillatory.

The following two lemmas will be needed in the sequel. The first one is contained in [5] and the second was given in a slightly less general form by Kartsatos and Onose in [8] (cf. also [6, Lemma 5.1]).

Lemma A. *Let $H: R_+ \times R_+ \to R_+\backslash\{0\}$ be continuous and increasing in the second variable, and assume further that the differential inequality*

$$x^{(n)} + H(t, x) \leqslant 0 \tag{1.2}$$

has a solution $z(t)$, $t \in [t_1, \infty)$, $t_1 \geqslant 0$, such that $z(t) > 0$, $t \geqslant t_1$. Then the equation

$$x^{(n)} + H(t, x) = 0 \tag{1.3}$$

has a positive solution $x(t)$ such that $x(t) \leqslant z(t)$ eventually.

Naturally, analogous results hold for eventually negative solutions of the inequality

$$x^{(n)} + H_1(t, x) \geqslant 0, \tag{1.4}$$

provided that $H_1: R_+ \times R_- \to R_-$ is continuous and increasing in its second variable.

Lemma B. *Consider the equation*

$$x^{(n)} + P(t, x, x', \ldots, x^{(n-1)})\, x^{(n-1)} + H(t, x) = 0 \tag{1.5}$$

with $H: R_+ \times R \to R$, continuous and such that $uH(t, u) > 0$ for every $u \neq 0$. Moreover, let $P: R_+ \times R^n \to R$ be continuous. Then if $x(t)$ is a nonoscillatory solution of (1.5) *we must have $x^{(n-1)}(t) > 0$ or $x^{(n-1)}(t) < 0$ for all large t. If,* *in addition, there exists a continuous function* $m: R_+ \to R$ *such that* $P(t, x_1, x_2, \ldots, x_n) \leqslant m(t)$ *for every* $(t, x_1, x_2, \ldots, x_n) \in R_+ \times R^n$ *and*

$$\lim_{t\to\infty} \int_{\bar{t}}^{t} \exp\left[-\int_{\bar{t}}^{u} m(s)\, ds\right] du = +\infty$$

for every $\bar{t} \geqslant 0$, *then* $x^{(n-1)}(t)\, x(t) > 0$ *for all large* t.

2. Comparison Results

In this section we consider equations of the form

$$[Q_1(t)\, x^{(n-1)}]' + H_1(t, x) = 0, \tag{2.1}$$

$$[Q_2(t)\, x^{(n-1)}]' + H_2(t, x) = 0, \tag{2.2}$$

and we show that, by comparison, the existence of certain nonoscillatory solutions for (2.1) implies the existence of the same type of solutions for (2.2). We then apply these results to equations of the form

$$x^{(n)} + P_1(t, x, x', \ldots, x^{(n-1)})\, x^{(n-1)} + H_1(t, x) = 0, \tag{2.3}$$

$$x^{(n)} + P_2(t)\, x^{(n-1)} + H_2(t, x) = 0 \tag{2.4}$$

and obtain, as a by-product, the oscillation of (2.3), provided (2.4) is oscillatory.

THEOREM 2.1. *Let the functions* $H_i(t, u)$, $i = 1, 2$, *be defined and continuous on* $R_+ \times R$ *with values in* R, *increasing w.r.t.* u, *and such that* $uH_i(t, u) > 0$ *for* $u \neq 0$. *Moreover, let* $[H_2(t, u) - H_1(t, u)] \operatorname{sgn} u \geqslant 0$ *for all* $(t, u) \in R_+ \times R$ *with* $u \neq 0$. *Let* $Q_i: R_+ \to R_+\backslash\{0\}$, $i = 1, 2$, *be continuous and satisfy* $Q_1(t) \geqslant Q_2(t)$, $t \in R_+$. *Then the existence of a positive (negative) solution* $x(t)$ *of* (2.2) *such that* $x^{(n-1)}(t) > 0$ $(x^{(n-1)}(t) < 0)$ *eventually, implies the same fact for* (2.1).

Proof. Assume first that $x(t)$ is a solution of (2.2) such that $x(t) > 0$, $x^{(n-1)}(t) > 0$ for $t \geqslant t_1 \geqslant 0$. Then all the derivatives of $x(t)$ up to and including the order $n - 2$ are of fixed sign for all large t. Moreover, there exists an odd integer k $(1 \leqslant k \leqslant n - 1)$ such that $(-1)^j x^{(j)}(t) < 0$ for $j = k + 1, k + 2, \ldots, n - 1$ and $x^{(j)}(t) > 0$ for $j = 1, 2, \ldots, k$ for all large t. We may (and do) assume that all these inequalities hold for all $t \geqslant t_1$. Now consider the transformation

$$u(t) \equiv Q_2(t)\, x^{(n-1)}(t) > 0, \qquad t \geqslant t_1. \tag{2.5}$$

Then dividing (2.5) by $Q_2(t)$, $t \geqslant t_1$, and integrating from t to v $(t_1 \leqslant t \leqslant v)$ we obtain

$$x^{(n-2)}(v) - x^{(n-2)}(t) = \int_t^v [u(s)/Q_2(s)]\, ds. \tag{2.6}$$

Thus, if $k = n - 1$, we obtain

$$x^{(n-2)}(v) \geqslant \int_t^v [u(s)/Q_2(s)]\, ds, \qquad v \geqslant t \geqslant t_1$$

and, in particular,

$$x^{(n-2)}(t) \geqslant \int_{t_1}^t [u(s)/Q_2(s)]\, ds, \qquad t \geqslant t_1 . \tag{2.7}$$

If $k < n - 1$, then $x^{(n-2)}(t) < 0$ and increasing in $[t_1, \infty)$. Thus, $\lim_{v\to\infty} x^{(n-2)}(v) = L \leqslant 0$, and from (2.6) we obtain

$$L - x^{(n-2)}(t) = \int_t^\infty [u(s)/Q_2(s)]\, ds, \qquad t \geqslant t_1 ,$$

which implies

$$x^{(n-2)}(t) \leqslant - \int_t^\infty [u(s)/Q_2(s)]\, ds, \qquad t \geqslant t_1 . \tag{2.8}$$

Repeating this process we get

$$x^{(k)}(t) \geqslant \int_t^\infty \int_{v_{n-k-2}}^\infty \cdots \int_{v_1}^\infty [u(s)/Q_2(s)]\, ds\, dv_1 \cdots dv_{n-k-2} \tag{2.9}$$

for all $t \geqslant t_1$. Now let $\phi_2(t; u)$ denote the right-hand member of (2.9), and integrate (2.9) k times to obtain

$$x(t) \geqslant x(t_1) + \int_{t_1}^t \int_{t_1}^{v_{k-1}} \cdots \int_{t_1}^{v_1} \phi_2(s; u)\, ds\, dv_1 \cdots dv_{k-1} . \tag{2.10}$$

This formula contains both cases: $k = n - 1$ and $k < n - 1$. Now we denote by $\Psi_2(t; u)$ the multiple integral above. Then from (2.2) we get

$$u'(t) + H_2(t, x(t_1) + \Psi_2(t; u)) \leqslant 0. \tag{2.11}$$

Now we use the fact that $Q_1(t) \geqslant Q_2(t)$, $\Psi_2(t; u) \geqslant \Psi_1(t; u)$ (where Ψ_1 is the same as Ψ_2 with H_1 replacing H_2), $t \geqslant t_1$ to obtain

$$u'(t) + H_1(t, x(t_1) + \Psi_1(t; u)) \leqslant 0. \tag{2.12}$$

Now since $x(t_1) + \Psi_1(t; u) > 0$ for $t \geqslant t_1$, (2.12) implies that $u'(t) < 0$ for $t \geqslant t_1$. Thus, $\lim_{t\to\infty} u(t) = \lambda$, where $0 \leqslant \lambda + \infty$. Thus, integrating (2.12) from t to v $(t_1 \leqslant t \leqslant v)$ and taking the limit of the resulting equation as $v \to +\infty$ we obtain

$$\lambda \leqslant u(t) - \int_t^\infty H_1(s, x(t_1) + \Psi_1(s; u))\, ds,$$

or

$$u(t) \geqslant \int_t^\infty H_1(s, x(t_1) + \Psi_1(s; u))\, ds. \tag{2.13}$$

Now consider the sequence

$$x_0(t) = u(t)$$
$$x_{k+1}(t) = \int_{t_1}^\infty H_1(s, x(t_1) + \Psi_1(s; x_k))\, ds,$$

$k = 0, 1,...$. Then it can be easily shown (cf. also Kartsatos [5, Lemma 2.1]) that

$$\begin{aligned} 0 < x_k(t) &\leqslant u(t), \qquad t - t_1, \quad k = 0, 1,... \\ x_{k+1}(t) &\leqslant x_k(t), \qquad t \geqslant t_1, \quad k = 0, 1,.... \end{aligned} \tag{2.14}$$

Thus, by Lebesgue's theorem of monotone convergence, there exists $\bar{x}(t)$ such that $\lim_{k\to\infty} x_k(t) = \bar{x}(t)$ and

$$\bar{x}(t) = \int_t^\infty H_1(s, x(t_1) + \Psi_1(s; \bar{x}))\, ds > 0 \tag{2.15}$$

for $t \geqslant t_1$. Differentiating $\bar{x}(t)$ we get

$$\bar{x}'(t) = -H_1(s, x(t_1) + \Psi_1(s; \bar{x})), \qquad t \geqslant t_1 .$$

Letting $z(t) \equiv x(t_1) + \Psi_1(t; \bar{x})$, we obtain $0 < z^{(n-1)}(t) = Q_1^{-1}(t)\, \bar{x}(t)$. Hence,

$$[Q_1(t)\, z^{(n-1)}(t)]' + H_1(t, z(t)) = 0, \qquad t \geqslant t_1 .$$

This completes the proof for a positive $x(t)$. Now let $x(t)$ be a solution of (2.2) such that $x(t) < 0$, $x^{(n-1)}(t) < 0$ for $t \geqslant t_1 \geqslant 0$. Then letting $v(t) \equiv -x(t)$, $t \geqslant t_1$, we obtain

$$[Q_2(t)\, v^{(n-1)}(t)]' - H_2(t, -v(t)) = 0. \tag{2.16}$$

Now the function $-H_2(t, -u)$ has exactly the same properties as $H_2(t, u)$. In particular, $-H_2(t, -v(t)) > 0$ for $t \geqslant t_1$ and $-H_2(t, -y) \geqslant -H_1(t, -y)$ for any $y > 0$. Thus, the above argument in the case $x(t) > 0$ can now be repeated to ensure the existence of a solution $z_1(t)$ of

$$[Q_1(t)\, z^{(n-1)}]' - H_1(t, -z) = 0 \tag{2.17}$$

such that $z_1^{(n-1)}(t) > 0$, $z_1(t) > 0$ for $t \geqslant t_1$. Letting $s(t) \equiv -z_1(t)$, $t \geqslant t_1$, we obtain the desired solution of (2.1). This completes the proof of the theorem.

COROLLARY 2.2. *Consider Eqs.* (2.3) *and* (2.4). *Let* $H_i: R_+ \times R \to R$, $i = 1, 2$, *be continuous, increasing in the second variable, and such that* $uH_i(t, u) > 0$ *for every* $u \neq 0$. *Let*

$$Q_1(t, T; u) = \exp\left(\int_T^t P_1(s, u(s),..., u^{(n-1)}(s))\, ds\right),$$

$$Q_2(t, T) = \exp\left(\int_T^t P_2(s)\, ds\right)$$

for every t, T *with* $0 \leqslant T \leqslant t$; *and every* $u \in C^{n-1}[T, \infty)$, *where* $P_1: R_+ \times R^n \to R$, $P_2: R_+ \to R$ *are both continuous and such that* $P_1(t, x_1, x_2,..., x_n) \leqslant P_2(t)$, $(t, x_1, x_2,..., x_n) \in R_+ \times R^n$. *Assume further that*

$$[Q_1(t, T; u)\, H_1(t, v) - Q_2(t, T)\, H_2(t, v)] \operatorname{sgn} v \geqslant 0$$

for every t, T *with* $t \geqslant T \geqslant 0$, *every* $u \in C^n[T, \infty)$, *and every* $v \in R$ *with* $v \neq 0$. *Then the existence of a positive* (*negative*) *solution* $x(t)$ *of* (2.3) *such that* $x^{(n-1)}(t) > 0$ ($x^{(n-1)}(t) < 0$), *eventually, implies the same fact for* (2.4).

Proof. It suffices to observe that if $x(t)$ is a solution of (2.3) such that $x(t) > 0$, $x^{(n-1)}(t) > 0$ for $t \geqslant \bar{t} \geqslant 0$, then $x(t)$ satisfies the equation

$$[Q_1(t, \bar{t}; x)\, x^{(n-1)}]' + Q_1(t, \bar{t}; x)\, H_1(t, x) = 0, \tag{2.18}$$

which is obtained from (2.3) by multiplication by $Q_1(t, \bar{t}; x)$. Also, (3.4) transforms into

$$[Q_2(t, \bar{t})\, x^{(n-1)}]' + Q_2(t, \bar{t})\, H(t, x) = 0, \qquad t \geqslant \bar{t}.$$

The proof now follows the steps of Theorem 2.1. Notice that Q_1H_1, Q_1 correspond to H_2, Q_2, respectively, of Theorem 2.1.

Now we have the following oscillation criterion:

COROLLARY 2.3. *Consider Eqs.* (2.3), (2.4) *under the assumptions of Corollary* 2.2. *Assume further the existence of a function* $m: R_+ \to R$, *continuous and such that*

$$P_1(t, x_1, x_2,..., x_n) \leqslant m(t), \qquad t \in R$$

for every $(t, x_1,..., x_n) \in R_+ \times R^n$ *and*

$$\lim_{t\to\infty} \int_{\bar{t}}^t \exp\left(-\int_{\bar{t}}^v m(s)\, ds\right) dv = +\infty$$

for every $\bar{t} \geqslant 0$. *Then if* (2.4) *is oscillatory*, (2.3) *is also oscillatory.*

Proof. It suffices to observe that if $x(t)$ is a nonoscillatory solution of (2.3) then, by Lemma B, $x(t)\, x^{(n-1)}(t) > 0$ for all large t. Thus, if $x(t) > 0$, $x^{(n-1)}(t) > 0$, $t \geqslant t_1 \geqslant 0$, then Corollary 2.2 ensures the existence of a positive solution of (2.4), a contradiction. One argues similarly in the case of a negative solution of (2.3).

The following example shows that $P_1 \leqslant P_2$ and $H_1 \geqslant H_2$ do not suffice for the oscillation of (2.3), although (2.4) is oscillatory and the rest of the assumptions of the above corollary hold.

$$x'' - 5x' + 4x = 0, \tag{2.3$_A$}$$

$$x'' + 2x' + 3x = 0. \tag{2.4$_A$}$$

We now establish a comparison theorem for solutions $x(t)$ such that $x(t)\, x^{(n-1)}(t) < 0$ eventually.

THEOREM 2.4. *Consider Eqs.* (2.3) *and* (2.4), *where* $P_1 \colon R_+ \times R^n \to R$, $P_2 \colon R_+ \to R$ *are continuous and such that* $P_1(t, x_1, x_2, ..., x_n) \leqslant P_2(t)$, $t \in R_+$, $(x_1, x_2, ..., x_n) \in R^n$. *Moreover, let* $H_i \colon R_+ \times R \to R$, $i = 1, 2$, *be continuous, increasing in the second variable, and such that* $uH_i(t, u) > 0$, *for* $u \neq 0$, $i \neq 1, 2$ *and*

$$[H_1(t, u) - H_2(t, u)] \operatorname{sgn} u \geqslant 0, \qquad t \in R_+, \qquad u \neq 0.$$

Then the existence of a positive (negative) solution $x(t)$ *of* (2.3) *such that* $x^{(n-1)}(t) < 0$ $(x^{(n-1)}(t) > 0)$ *eventually, implies the same fact for* (2.4).

Proof. Let $x(t)$ be a solution of (2.3) such that $x(t) > 0$, $x^{(n-1)}(t) < 0$ for $t \geqslant t_1 \geqslant 0$. Then there exists an even integer k $(0 \leqslant k \leqslant n-2)$ such that $x^{(j)}(t) > 0$ for $j = 0, 1, ..., k$ and $(-1)^j x^{(j)}(t) > 0$ for $j = k+1, k+2, ..., n-1$, and for all large t. We assume that all these inequalities hold for $t \geqslant t_1$. Now the function $x(t) \equiv x^{(n-1)}(t)$, $t \geqslant t_1$, satisfies the first-order equation

$$u' + P_1(t, x(t), ..., x^{(n-1)}(t))u + H_1(t, x(t)) = 0. \tag{2.19}$$

Solving this linear equation, we obtain

$$\begin{aligned} x^{(n-1)}(t) &= \exp\left(-\int_{t_1}^{t} \bar{P}_1(s)\, ds\right)\left[x^{(n-1)}(t_1) - \int_{t_1}^{t} H_1(s, x(s)) \exp\left(\int_{t_1}^{s} \bar{P}_1(v)\, dv\right) ds\right] \\ &\leqslant \exp\left(-\int_{t_1}^{t} P_2(s)\, ds\right)\left[x^{(n-1)}(t_1) - \int_{t_1}^{t} H_2(s, x(s)) \exp\left(\int_{t_1}^{s} P_2(v)\, dv\right) ds\right] \end{aligned}$$

$$\left(\int_t^s P_2(v)\, dv \leqslant \int_t^s \bar{P}_1(v)\, dv,\ t \geqslant v\right) \tag{2.20}$$

for all $t \geqslant t_1$, where $\bar{P}_1(t) \equiv P_1(t, x(t), \ldots, x^{(n-1)}(t))$. Integrating (2.20) from t to u $(t_1 \leqslant t \leqslant u)$ and denoting its last member by $\phi_2(t; x)$, we obtain

$$x^{(n-2)}(u) - x^{(n-2)}(t) \leqslant \int_t^u \phi_2(s; x)\, ds. \tag{2.21}$$

Since $\phi_2(t; x) < 0$ for $t \geqslant t_1$, the integral above has a limit as $u \to +\infty$ which cannot be $-\infty$ because this would imply $\lim_{t\to\infty} x^{(n-2)}(u) = -\infty$, a contradiction to the positiveness of $x(t)$. Thus,

$$x^{(n-2)}(\infty) - x^{(n-2)}(t) \leqslant \int_t^\infty \phi_2(s; x)\, ds, \tag{2.22}$$

where $x^{(n-2)}(\infty) = \lim_{u\to\infty} x^{(n-2)}(u) \geqslant 0$. Consequently,

$$x^{(n-2)}(t) \geqslant -\int_t^\infty \phi_2(s; x)\, ds. \tag{2.23}$$

Continuing in the same way, we obtain

$$x^{(k)}(t) \geqslant -\int_t^\infty \int_{v_{n-k-2}}^\infty \cdots \int_{v_1}^\infty \phi_2(s; x)\, ds\, dv_1 \cdots dv_{n-k-2} \tag{2.24}$$

for every $t \geqslant t_1$. Integrating k times from t_1 to $t \geqslant t_1$ we find

$$x(t) \geqslant \int_{t_1}^t \int_{t_1}^{v_{k-1}} \int_{t_1}^{v} \phi_2^*(s; x)\, ds\, dv_1 \cdots dv_{k-1} \tag{2.25}$$

where $\phi_2^*(t; x)$ denotes the second member of (2.24). Now the proof goes exactly as in Theorem 2.1 (starting from (2.13)), because the functional $\phi_2^*(s; u)$ is positive and increasing in u. Thus, we obtain in this way a nonnegative solution $\bar{x}(t)$ satisfying the integral equation

$$\bar{x}(t) = \int_{t_1}^t \int_{t_1}^{v_{k-1}} \cdots \int_{t_1}^{v_1} \phi_2^*(s; \bar{x})\, ds\, dv_1 \cdots dv_{k-1}.$$

Obviously, $\bar{x}(t) > 0$ for $t > t_1$ because $\bar{x}^{(k)}(t) = \phi_2^*(t; \bar{x}) > 0$, $t \geqslant t_1$. Moreover, we have

$$\bar{x}^{(n-1)}(t) = \exp\left(-\int_{t_1}^t P_2(s)\, ds\right)\left[x^{(n-1)}(t_1) - \int_{t_1}^t H_2(s, \bar{x}(s)) \exp\left(\int_{t_1}^s P_1(u)\, du\right) ds\right].$$

Thus, $\bar{x}(t)$, $t \geqslant t_1$ is a solution to (2.4) with the desired properties. The case $x(t) < 0$, $x^{(n-1)}(t) > 0$ is easily covered, as in the proof of the corresponding case of Theorem 2.1, by the transformation $u(t) \equiv -x(t)$, $t \geqslant t_1$. This completes the proof of the theorem.

In the following corollary we provide conditions on P_1, H_1 so that every nonoscillatory solution of (2.3) is actually of the form in the statement of the above theorem. Thus, we can obtain an oscillation result if we assume further that (2.4) is oscillatory.

COROLLARY 2.5. *Let the assumptions of Theorem* 2.4 *be satisfied along with the either one of the following*:

(i) $P_1(t, x_1, x_2, \ldots, x_n) \geqslant 0$ *for* $(t, x_1, x_2, \ldots, x_n) \in R_+ \times R^n$ *and the equation* $x^{(n)} + H_1(t, x) = 0$ *is oscillatory.*

(ii) *for every* $t_1 \geqslant 0$, *every* $u \in C^{n-1}[t_1, \infty)$, *and every* $k > 0$,

$$\liminf_{t\to\infty} \left[\int_{t_1}^{t} H_1(s, k)\, Q_u(s)\, ds\right] \Big/ Q_u(t) > 0, \tag{2.27}$$

$$\limsup_{t\to\infty} \left[\int_{t_1}^{t} H_1(s, -k)\, Q_u(s)\, ds\right] \Big/ Q_u(t) < 0, \tag{2.28}$$

where $Q_u(t) \equiv \exp\{\int_{t_1}^{t} P_1(s, u(s), u'(s), \ldots, u^{(n-1)}(s))ds\}$. *Moreover, let* $\lim_{t\to\infty} Q_u(t) = +\infty$, $u \in C^{n-1}[t_1, +\infty)$, $t_1 \geqslant 0$.

Then if (2.4) *is oscillatory*, (2.3) *is also oscillatory.*

Proof. Let (i) be satisfied and assume that $x(t)$ is a solution of (2.3) such that $x(t) > 0$ and $x^{(n-1)}(t) > 0$ for $t \geqslant t_1 \geqslant 0$. Then from (2.3) we obtain $x^{(n)}(t) + H_1(t, x(t)) \leqslant 0$. Now applying Lemma A, we get a positive solution to the equation $u^{(n)} + H(t, u) = 0$, a contradiction. Thus, if $x(t) > 0$ eventually, $x^{(n-1)}(t) < 0$ eventually. Theorem 2.4 applies now to ensure the existence of a positive solution to Eq. (2.4), a contradiction. Similarly for a negative solution $x(t)$ of (2.3). Now let (2.27) be satisfied and let $x(t)$ be a solution of (2.3) such that $x(t) > 0$, $t \geqslant t_1 \geqslant 0$. Then $x^{(n-1)}(t) \equiv u(t)$ satisfies Eq. (2.19) for $t \geqslant t_1$. Solving (2.19) we obtain (2.20). Now we know that all the derivatives of $x(t)$ up to and including order $n - 1$ are of fixed sign (positive or negative) for all large t, say for $t \geqslant t_1$ (Lemma B). If $x^{(n-1)}(t) > 0$ for $t \geqslant t_1$, then since n is even $x'(t) > 0$ for $t \geqslant t_1$, and $x(t) \geqslant x(t_1) > 0$ for $t \geqslant t_1$. Thus, from (2.20) we obtain

$$\begin{aligned} x^{(n-1)}(t) &= \left[x^{(n-1)}(t_1) - \int_{t_1}^{t} H_1(s, x(s))\, Q_x(s)\, ds\right] \Big/ Q_x(t) \\ &\leqslant \left[x^{(n-1)}(t_1) - \int_{t_1}^{t} H_1(s, x(t_1))\, Q_x(s)\, ds\right] \Big/ Q_x(t), \end{aligned} \tag{2.29}$$

which, along with (2.27), implies that $x^{(n-1)}(t) < 0$ for all large t, a contradiction. Thus, $x^{(n-1)}(t) < 0$ for all large t, and the conclusion of this corollary follows as above from Theorem 2.4.

COROLLARY 2.6. *Under the hypotheses of Theorem* 2.4, *but with the inequality on* H_1, H_2 *replaced by*

$$[H_1(t, u) - H_2(t, u)] \operatorname{sgn} u \geqslant 0, \qquad t \in R_+, \quad 0 < |u| < \delta$$

(δ *some positive constant*), *assume further that* (2.4) *is oscillatory. Then every solution of* (2.3) *which tends to zero as* $t \to +\infty$ *is oscillatory.*

Proof. It suffices to observe that if $x(t)$ is a positive solution of (2.3) such that $\lim_{t\to\infty} x(t) = 0$, then, since n is even, this is possible only if $x^{(n-1)}(t) < 0$. Thus, choosing $t_1 \geqslant 0$ such that $x(t) < \delta$ and $x^{(n-1)}(t) < 0$ for $t \geqslant t_1$, and applying Theorem 2.4 we obtain a contradiction to the oscillation of (2.4).

3. APPLICATIONS

In this section we establish some applications of Theorem 2.4 and Corollary 2.6 to equations of the form

$$x'' + f(t, x, x')\, x' + g(t, x) = 0 \tag{3.1}$$

by using as a comparison equation

$$x'' + kx' + G(x) = 0, \tag{3.2}$$

where k is a positive constant. The form (3.1) contains several generalized forms of the equation of Lienard and the equation of Van der Pol. Of course it is to be understood that the second-order results which follow depend heavily on the comparison equation (3.2) and are mainly presented here to illustrate the power of the results in Section 2. Depending on the particular equation of the form (3.1) under consideration, it might be advisable to choose comparison equations more suitable than (3.2).

We first provide conditions below under which (3.2) is oscillatory. Then we obtain by comparison the oscillation of (3.1). The function G in (3.2) is assumed to be of the sublinear type, i.e., if $G(u) = |u|^\alpha \operatorname{sgn} u$, then $0 < \alpha < 1$.

LEMMA 3.1. *Let* G *in* (3.2) *satisfy the following assumptions*: $G\colon R \to R$, *continuous, increasing, and such that* $uG(u) > 0$ *for* $u \neq 0$. *Moreover, let*

$$\int_{0+}^{\epsilon} \frac{du}{G(u)} < +\infty, \qquad \int_{0-}^{-\epsilon} \frac{du}{G(u)} < +\infty$$

for every $\epsilon > 0$.

Then every solution of (3.2) *is oscillatory.*

Proof. Assume the existence of a solution $x(t)$ of (3.2) such that $x(t) > 0$, $t \geqslant t_1$. Then from Lemma B we obtain that $x'(t) \gtrless 0$ eventually. Let $x'(t) > 0$, $t \geqslant t_1$. Then we have

$$x''(t) + G(x(t)) \leqslant 0, \qquad t \geqslant t_1 .$$

Thus, by Lemma A, the equation

$$u'' + G(u) = 0$$

has a positive solution, but this is impossible by well-known results. Thus, $x'(t) < 0$ eventually. If we assume that $x(t) \geqslant \mu$, $x'(t) < 0$ for $t \geqslant t_1$, with μ a positive constant, then one integration of (3.2) from t_1 to $t \geqslant t_1$ and the use of the inequality $G(x(t)) \geqslant G(\mu)$ for $t \geqslant t_1$ imply that $\lim_{t\to\infty} x'(t) = -\infty$, a contradiction to the positiveness of $x(t)$. Thus, $\lim_{t\to\infty} x(t) = 0$. Now we integrate (3.2) from t to $u \geqslant t \geqslant t_1$ to obtain

$$x'(u) - x'(t) + k(x(u) - x(t)) = -\int_t^u G(x(s))\, ds. \tag{3.3}$$

Since $G(x(t)) > 0$ for $t \geqslant t_1$, the limit of the integral above exists as $u \to +\infty$ and is finite or $+\infty$. The second possibility cannot happen because it implies $\lim_{u\to\infty} x'(u) = -\infty$, a contradiction. Thus, $\lim_{u\to\infty} x'(u) = M \leqslant 0$. If $M < 0$, then $x(t) \to -\infty$ as $t \to +\infty$, a contradiction. Consequently, $M = 0$ and taking limits as $u \to +\infty$ in (3.3) we obtain

$$kx(t) \geqslant x'(t) + kx(t) = \int_t^\infty G(x(s))\, ds, \qquad t \geqslant t_1 . \tag{3.4}$$

Now we divide by $\int_t^\infty G(x(s))\, ds > 0$ to obtain $x(t)/u(t) \geqslant 1$, $t \geqslant t_1$, where $u(t)$ is the above integral divided by $k > 0$. Since G is increasing, it follows that $G(x(t))/G(u(t)) \geqslant 1$ for every $t \geqslant t_1$. Thus, one integration of this inequality from t_1 to $t \geqslant t_1$ yields

$$-k \int_{t_1}^t \frac{u'(s)\, ds}{G(u(s))} = -k \int_{u(t_1)}^{u(t)} \frac{dv}{G(v)} \geqslant t - t_1 . \tag{3.5}$$

Now $\lim_{t\to\infty} u(t) = 0$ and (3.5) implies a contradiction to the integral condition assumed on G. A similar proof holds in the case of a negative solution. This completes the proof.

THEOREM 3.2. *Consider* (3.1) *under the following assumptions*:

(i) $f: R_+ \times R^2 \to R_+$, *continuous and such that* $f(t_1, x_1, x_2) \leqslant k$ *for all* $(t_1, x_1, x_2) \in R_+ \times R^2$.

(ii) $g: R_+ \times R \to R$, *continuous, increasing in the second variable, and such that*

$$[g(t, u) - G(u)] \operatorname{sgn} u \geqslant 0 \qquad \text{for} \quad u \neq 0, \quad t \in R_+,$$

where G satisfies the hypotheses of Lemma 3.1. *Then* (3.1) *is oscillatory.*

Proof. To apply Theorem 2.4, it suffices to show that $x'(t) < 0$ ($x'(t) > 0$) eventually, for every positive (negative) solution of (3.1). As before, we present only the proof for a positive solution of (3.1). In fact, let $x(t)$ solve (3.1) and assume that $x(t) > 0$, $x'(t) > 0$ for every $t \geqslant t_1 \geqslant 0$. Then from (3.1) we obtain

$$x''(t) + G(x(t)) \leqslant x''(t) + f(t, x(t), x'(t))\, x'(t) + g(t, x(t)) = 0$$

for all $t \geqslant t_1$. Thus, by Lemma A, the equation $u'' + G(u) = 0$ has a positive solution, a contradiction. This completes the proof.

THEOREM 3.3. *For Eq.* (3.1) *assume the following*:

(i) $F(t, x_1, x_2) \equiv F(x_1, x_2)$, $F: R^2 \to R_+$, *continuous and such that* $\lim_{x_1 \to 0} F(x_1, x_2) = 0$ *uniformly in* x_2;

(ii) $g(t, u) \equiv g(u)$ *with* $g: R \to R$ *continuous and*

$$\int_0^\infty g(u)\, du = +\infty.$$

Moreover, there exists $\delta > 0$ *such that for all* $u \in R$ *with* $0 < |u| < \delta$, *we have* $[g(u) - G(u)] \operatorname{sgn} u \geqslant 0$. *Here G satisfies the hypotheses of Lemma* 3.1. *Then* (3.1) *is oscillatory.*

Proof. From Theorem 1 of Utz [11] it follows that every nonoscillatory solution of (3.1) tends monotonically to zero as $t \to +\infty$. Thus Corollary 2.6 is directly applicable to yield the conclusion.

4. FORCED EQUATIONS

As an application of the methods developed in Theorems 2.1 and 2.4 we consider the oscillation of a forced equation of the form

$$x^{(n)} + P(t)\, x^{(n-1)} + H(t, x) = Q(t). \tag{4.1}$$

THEOREM 4.1. *In Eq.* (4.1) *let* $P: R_+ \to R$, $H: R_+ \times R \to R$, *and* $Q: R_+ \to R$ *be continuous. In addition, let* $uH(t, u) > 0$ *for* $u \neq 0$ *and* H *be increasing in* u.

Moreover let the function $S: R_+ \to R$ be such that $S^{(n)}(t) + P(t)\, S^{(n-1)}(t) \equiv Q(t)$, $t \in R_+$ and $\lim_{t\to\infty} S(t) = 0$. Then if every solution of the equation

$$x^{(n)} + P(t)\, x^{(n-1)} + H(t, x) = 0 \tag{4.2}$$

is oscillatory or tends monotonically to zero, the same property is shared by Eq. (4.1).

Proof. Assume that $x(t)$ is a nonoscillatory solution of (4.1) such that $x(t) > 0$ for every $t \geqslant t_1 \geqslant 0$. Then the function $u(t) \equiv x(t) - S(t)$, $t \geqslant t_1$, satisfies the equation

$$u^{(n)} + P(t)\, u^{(n-1)} + H(t, u + S(t)) = 0. \tag{4.3}$$

Now Lemma B ensures that $u^{(n-1)}(t) \lesseqgtr 0$ for all large t because $u(t) + S(t) = x(t) > 0$. Let $u^{(n-1)}(t) > 0$, $u(t) > 0$ for $t \geqslant t_2 \geqslant t_1$. Then since n is even $u'(t) > 0$ for $t \geqslant t_3 \geqslant t_2$. Thus, $u(t) \geqslant u(t_3) > 0$ for $t \geqslant t_3$. Now choose ϵ and $t_4 \geqslant t_3$ such that $0 < \epsilon < u(t_3)$ and $|S(t)| < \epsilon$ for $t \geqslant t_4$. Then we have $H(t, u(t) + S(t)) \geqslant H(t, u(t) - \epsilon) > 0$ for $t \geqslant t_4$. Now we apply the argument of Theorem 2.1 to obtain a positive solution $v(t)$, $t \geqslant t_4$, to the equation

$$v^{(n)} + P(t)\, v^{(n-1)} + H(t, v - \epsilon) = 0. \tag{4.4}$$

This function $v(t)$ actually satisfies $v(t_4) = u(t_4) > \epsilon$, and $v(t) \geqslant v(t_4)$ for all $t \geqslant t_4$. A closer look at the proof of Theorem 2.1 reveals this fact. Now if we let $z(t) \equiv v(t) - \epsilon > 0$, $t \geqslant t_4$, we obtain a positive solution $z(t)$ to Eq. (4.2), a contradiction to its assumed oscillation. Now let $u^{(n-1)}(t) < 0$ and $u(t) > 0$ eventually. Thus either $u'(t) < 0$ or $u'(t) > 0$ for all large t. Let $u'(t) < 0$ for all large t, say for $t \geqslant t_2 \geqslant t_1$, and let the inequalities preceding (2.19) hold for all $t \geqslant t_3 \geqslant t_2$. Now assume that $\lim_{t\to\infty} u(t) = \mu > 0$, and choose ϵ such that $0 < \epsilon < \mu$, and $|S(t)| < \epsilon$ for all $t \geqslant t_4 \geqslant t_3$.

Then since $u(t)$ is decreasing, $u(t) + S(t) \geqslant \mu + S(t) \geqslant \mu - \epsilon > 0$ for every $t \geqslant t_4$. Since the even integer k equals zero at the beginning of the proof of Theorem 2.4 (with x replaced by u) we obtain

$$u(t) \geqslant \mu - \int_t^\infty \int_{v_{n-2}}^\infty \cdots \int_{v_1}^\infty \phi^*(s; u)\, ds\, dv_1 \cdots dv_{n-2} \tag{4.5}$$

for all $t \geqslant t_4$, where ϕ^* is as in (2.25) but with t_4 instead of t_1, $u(t) + S(t)$ replacing $x(t)$ in the argument of H_2, H replacing H_2, and $u^{(n-1)}(t_4)$ replacing $x^{(n-1)}(t_1)$. Thus, as in the proof of Theorem 2.4, we obtain a solution $v(t)$ to the integral equation of (4.5) with $\phi^*(s; u - \epsilon)$ instead of $\phi^*(s; u)$, which satisfies $\lim_{t\to\infty} v(t) = \mu > \epsilon$.

This function $v(t)$ satisfies the equation

$$v^{(n)} + P(t)\, v^{(n-1)} + H(t, v - \epsilon) = 0. \tag{4.6}$$

Setting $w(t) = v(t) - \epsilon > 0$, we obtain a positive solution to (4.2), a contradiction.

If $u'(t) > 0$ for $t \geqslant t_3$, then $k \geqslant 2$ and instead of (2.25) we may consider

$$u(t) \geqslant u(t_3) + \int_{t_3}^{t} \int_{t_3}^{v_{k-1}} \cdots \int_{t_3}^{v} \phi_1^*(s; u)\, ds\, dv_1 \cdots dv_{k-1}, \tag{4.7}$$

with ϕ_1^* obtained from ϕ^* in (4.5) as in (2.25). Again we may take $0 < \epsilon < u(t_3)$ and $|S(t)| < \epsilon$ for $t \geqslant t_3$, in which case we have $H(t, u(t) + S(t)) \geqslant H(t, u(t) - \epsilon) > 0$ for $t \geqslant t_3$. Thus, we may again apply the approximation process in (4.7) with $\phi_1^*(s; u - \epsilon)$ instead of $\phi_1^*(s; u)$ to obtain a solution $v(t)$ of the differential equation (4.6) such that $v(t) - \epsilon \geqslant v(t_3) - \epsilon = u(t_3) - \epsilon$ for $t \geqslant t_3$. Setting $w(t) = v(t) - \epsilon$, $t \geqslant t_3$ we obtain a contradiction as before. Thus, if $u(t)$ is positive it must satisfy $\lim_{t\to\infty} u(t) = 0$. If $u(t)$ is negative, then $x(t) < S(t)$ for all large t, which implies $\lim_{t\to\infty} x(t) = 0$. Consequently, the conclusion of the theorem is true for positive solutions of (4.1). A similar proof covers the case of a negative $x(t)$. This completes the proof.

COROLLARY 4.2. *In addition to the hypotheses made in Theorem* 4.1, *let* $S(t)$ *be oscillatory and assume the following condition:*

(i) $P(t) \leqslant m(t)$ *for every* $t \in R_+$, *where* $m: R_+ \to R$ *is continuous and such that for every* $\bar{t} \geqslant 0$,

$$\lim_{t\to\infty} \int_{\bar{t}}^{t} \exp\left[-\int_{\bar{t}}^{u} m(s)\, ds\right] du = +\infty;$$

Then (4.1) *is oscillatory if* (4.2) *is oscillatory.*

Proof. It suffices to observe that if $x(t)$ is a solution of (4.1) such that $x(t) > 0$ for $t \geqslant t_1 \geqslant 0$, then the function $u(t)$ in (4.3) is either negative or positive with $u^{(n-1)}(t) > 0$ eventually. The case of $u(t)$ negative cannot happen now because $S(t)$ is oscillatory. Moreover, if $u(t) > 0$, $u^{(n-1)}(t) > 0$, then, since n is even, $u'(t) > 0$ and $\lim_{t\to\infty} u(t) = 0$ is impossible. Thus, all cases about $u(t)$ are impossible in the proof of Theorem 4.1. Consequently, $u(t)$ does not exist and $x(t)$ cannot be positive. Similarly, $x(t)$ cannot be eventually negative, and this completes the proof.

5. DISCUSSION

It should be mentioned here that all the above results hold for bounded solutions of the equations considered if the words "oscillation" and "oscillatory" are replaced everywhere by the words "B-oscillation" and "B-oscillatory," respectively, and the word "solution" is replaced by the word "bounded solution." Also, the results of the present paper are extendable to equations with n odd as well as functional equations of the form

$$x^{(n)} + P(t, x, x', \ldots, x^{(n-1)})\, x^{(n-1)} + H(t, x(g(t))) = 0,$$

with $g(t)$ continuous, increasing, and such that $\lim_{t\to\infty} g(t) = +\infty$. For a comparison theorem concerning equations of the above type with $P \equiv 0$, the reader is referred to the paper [9] of Kartsatos and Onose.

The main difference between the comparison results in this paper and those of Svec [10] and Bobisud [3] is that, as far as oscillation criteria are concerned, Svec and Bobisud have used linear or linearized comparison equations for all or some values of the dependent variable. The definite advantage in the present work is that a nonlinear comparison equation may satisfy weaker conditions ensuring its oscillation than those of a corresponding linear equation. For example for $n = 2$, interesting comparison equations in the present setting can be those considered in Theorem 2.8 of Erbe [4], or various equations in the paper [11] of Utz. As far as the present authors know, no comparison theorem is known for equations with middle terms multiplied by $x^{(n-2)}(t)$ instead $x^{(n-1)}(t)$. Some oscillation results for such equations are given in [7]. It would be interesting to allow perturbations Q in Theorem 4.1 depending also on x. For such perturbations but for the oscillation of all bounded solutions of (4.1) with $P \equiv 0$, the reader is referred to [6, Theorem 4.1]. In Theorems 2.1 and 2.4 the functions H_1, H_2 to be compared may contain higher-order derivatives of the dependent variable, provided the properties w.r.t. the second variable are maintained. However, the above possibility for the comparison equations is an open problem.

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